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Meshless Methods for Numerical Solutions of Inverse Problems

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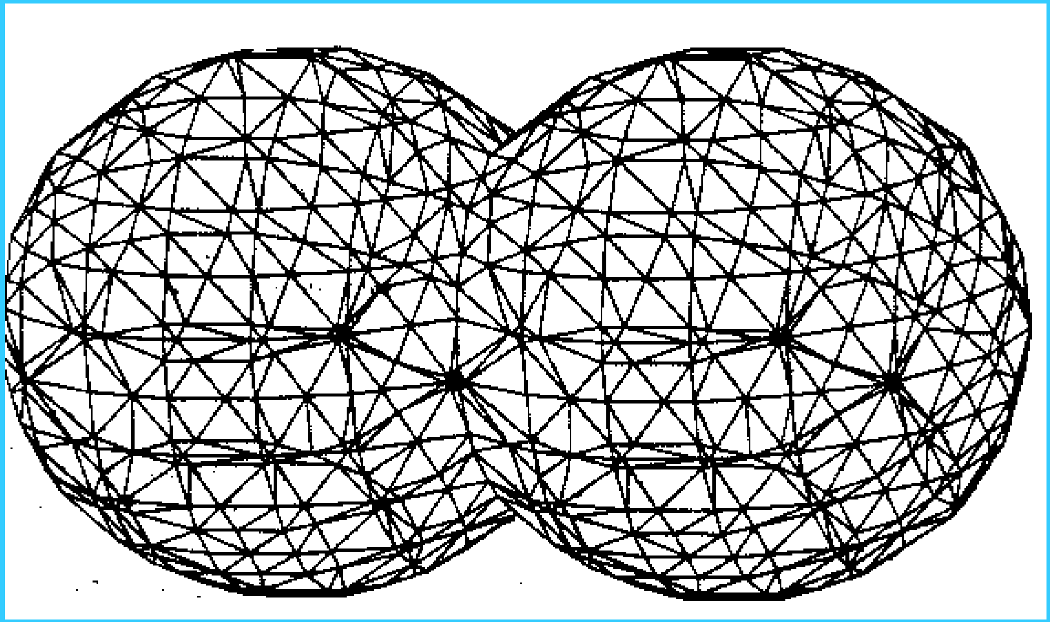
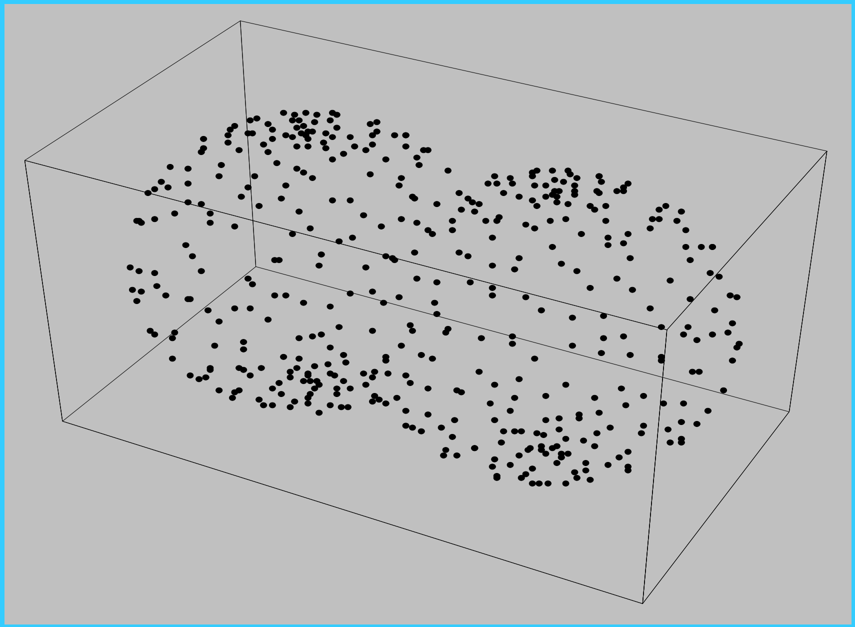
OBJECTIVE

THE METHOD OF FUNDAMENTAL SOLUTION IS COMBINED WITH THE RADIAL BASIS FUNCTIONS AS A TRULY '**MESH-LESS**' METHOD FOR THE NUMERICAL SOLUTIONS OF INVERSE PROBLEMS

MESH-DEPENDENT METHOD



MESHLESS METHOD



Advantages of Meshless Methods

- It requires neither domain nor surface discretization.
- The method is good for higher dimensional problems.
- It does not involve numerical integration.
- Ease to program.
- Cost effectiveness.

Two inverse problems in non-destructive testing

- Cauchy problem of elliptic equation

$$\begin{cases} Lu = 0 & in\ \Omega \\ u = f & on\ \Gamma \\ \dfrac{\partial u}{\partial n} = g & on\ \Gamma \end{cases}$$

- Inverse heat conduction problem

$$u_t - \Delta u = 0,\ in\ D$$
$$u|_{t=0} = \varphi$$
$$u|_{\Gamma} = f$$
$$\frac{\partial u}{\partial n}|_{\Gamma} = g$$
$$D = \Omega \times (0, t_{\max})$$

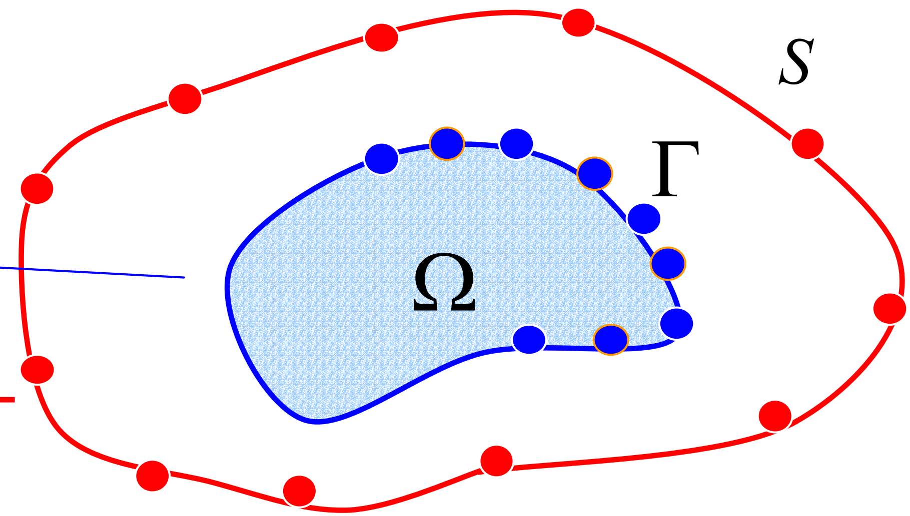
MFS

Consider a Cauchy problem for elliptic operator

$$\begin{cases} Lu = 0 & in \ \Omega \\ u = f & on \ \Gamma \\ \dfrac{\partial u}{\partial n} = g & on \ \Gamma \end{cases}$$

- Let $G(P,Q)$ be a fundamental solution for elliptic operator L .
- Choose a surface S containing Ω in its interior and k points $\{Q_j\}_{j=1}^{k}$ on S.
- Approximate u by

$$u_k = \sum_{j=1}^{k} \lambda_j G(P, Q_j)$$



$n+m$ points on the boundary

k source points

COLLOCATION

Choose collocation points $\{P_i\}_{i=1}^{n+m}$ on Γ and set

$$\sum_{j=1}^{k} \lambda_j G(P_i, Q_j) = f(P_i), \qquad 1 \le i \le n,$$

$$\sum_{j=1}^{k} \lambda_j \frac{\partial G(P_i, Q_j)}{\partial n} = g(P_i), \qquad n+1 \le i \le n+m.$$

then solve a least-squares problem $\min\limits_{\lambda} \| A\lambda - b \|$

$$A = \begin{pmatrix} G(P_i, Q_j) \\ \dfrac{\partial G(P_i, Q_j)}{\partial n} \end{pmatrix} \quad \textit{and} \quad b = \begin{pmatrix} f(P_i) \\ g(P_i) \end{pmatrix}$$

Note : A major concern with this method is the ill-conditioning of matrix A .

Tikhonov regularization: *Regularized solution* λ_α *is given by solving*

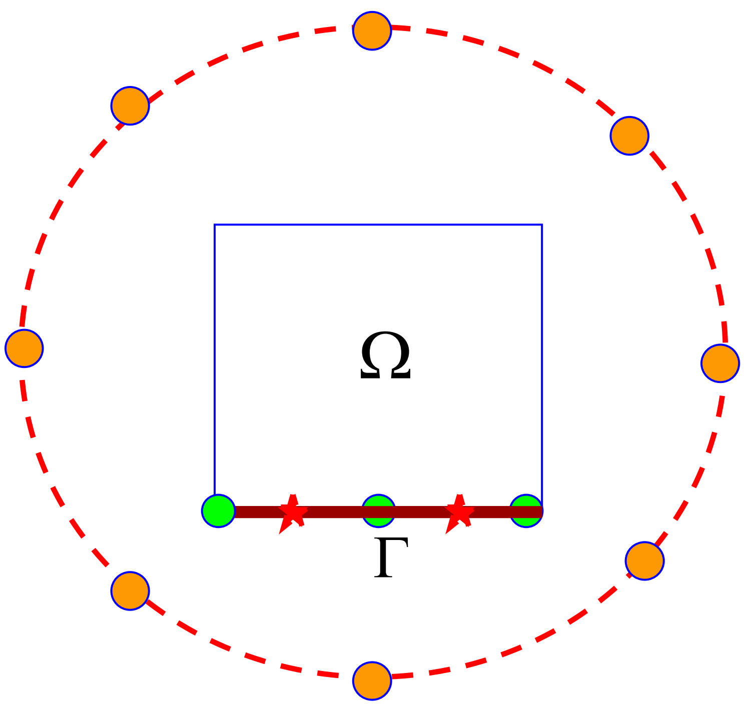
$$\min_{\lambda} \{ \| A\lambda - b \|^2 + \alpha^2 \| \lambda \|^2 \}$$

The choice of regularization parameter

- Generalized Cross Validation method
- L-curve method

Example 1. Cauchy problem in 2D

$$\Delta u(x,y) = 0, \qquad (x,y) \in \Omega,$$
$$u(x,y) = x^3 - 3xy^2 + e^{2y}\sin(2x) - e^x\cos(y).$$



★ - - - Collocation points for Neumann condition

● - - - Collocation points for Dirichlet condition

● - - - Source points

Example 2. ***L*= in 2D**

$$\Delta u + 100\,u = 0$$

$$u(x, y) = e^{5\sqrt{2}(x+y)i}$$

Example 3. ***L*= in 2D**

$$\Delta u - (2\pi)^2 u = 0$$

$$u(x, y) = \frac{1}{e^{2\sqrt{2}\pi}} \sin(2\pi x)\, e^{2\sqrt{2}\pi y}$$

Numerical results for examples 1, 2 , 3

By Tikhonov regularization and GCV choice of regularization parameter

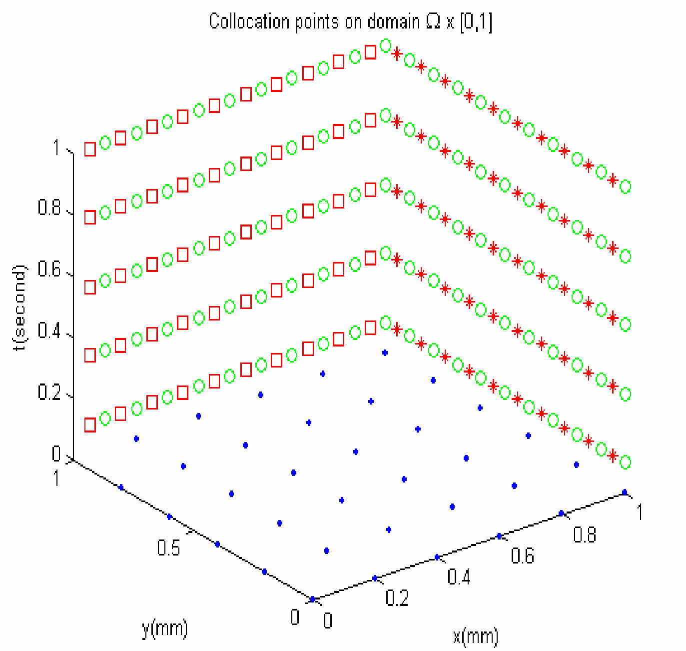
	RMS error of $u, \nabla u$	
	exact input data	noisy input data
Example 1	3.5475E-5 , 4.9007E-4	5.3702E-2 , 4.5011E-1*
Example 2	4.0282E-5 , 8.0576E-4	5.1200E-2 , 2.0373E-1**
Example 3	8.0523E-7 , 1.3483E-5	1.2679E-2 , 9.8170E-2*

* error level is 1E-3, **error level is 1E-4

Example 4. Inverse heat conduction problem

$$u_t(x,y,t) = \Delta u(x,y,t), \qquad (x,y,t) \in D$$
$$u(x,y,t) = e^{-4t}(\cos(2x + \cos(2y)).$$

$$D = \Omega \times (0,1)$$
$$\Omega = \{(x,y) \mid 0 < x, y < 1\}$$
$$\Gamma = \overline{\Omega} \cup \{x = 1\} \cup \{y = 1\}$$



Publication

[1] A fundamental solution method for inverse heat conduction problem, Engineering Analysis with Boundary Elements, (2003) in press.

[2] Computation for multi-dimensional Cauchy problem, SIAM Journal on Control and Optimization, 42(2003), 381-396.

[3] An orthonormal basis functions method for moment problem, Engineering Analysis with Boundary Elements, 26(2002), 855-860.

[4] Backus-Gilbert algorithm for the Cauchy problem of the Laplace equation, Inverse Problems, 17(2001), 261-271 .

[5] Numerical computation of a Cauchy problem for Laplace‘ equation , ZAMM Z. Angew. Math. Mech., 81(2001), no.10, 665-674.